

# Robust Natural Control of Distributed Systems

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The effect of having both parameter uncertainties and errors arising from spatial discretization on the performance of control systems is analyzed for distributed structures. It is shown that having errors in the system parameters, as well as errors due to a finite element type of discretization, is the equivalent to implementing the control laws by a set of admissible functions instead of by the actual eigenfunctions. It is shown that if natural control is considered in conjunction with modal filters and an adequate number of sensors, implementation of the control action by admissible functions leads to a stable closed-loop system. Using Gerschgorin's disks, the deviation of the desired closed-loop eigenvalues is determined.

## I. Introduction

ONE question that arises in the control of distributed-parameter systems is the amount of accuracy with which the model parameters are known and how this accuracy affects the performance of the control system. Most of the techniques proposed for the control of distributed systems are modal control methods,<sup>1-3</sup> so that their implementation requires accurate knowledge of the eigenvalues and eigenfunctions describing the motion. Design of the control system is based on the assumption that these eigenquantities are known precisely.

The motion of a distributed system such as a flexible spacecraft is governed by coupled sets of partial differential equations.<sup>4</sup> One may not know the parameters contained in the equations of motion and some of the boundary conditions accurately. Even if these parameters are known accurately, a closed-form solution to the distributed eigenvalue problem for the structure is generally not available because of the complexity of the distributed system. In such cases, an approximate solution is obtained by discretizing the distributed eigenvalue problem in space. Discretization methods that have enjoyed popularity include the Galerkin<sup>4</sup> and finite element<sup>5</sup> methods. Upon discretization, one obtains estimates of the eigenvalues and corresponding eigenfunctions that are subsequently used in the control system design.

Many different modal control techniques have been proposed for the control of large flexible structures.<sup>6-8</sup> However, most of these techniques encounter problems as the order of the control system is increased. Indeed, as the number of controlled modes gets larger, one experiences tremendous computational difficulties in determining the feedback gains.<sup>3</sup> An approach capable of handling large-order systems is natural control, which is also known as the independent modal-space control (IMSC) method.<sup>8-10</sup> Using natural control, artificial damping is provided to each mode of the structure independent of the other modes, so that the control system design and implementation become very simple. Natural controls represent methods that preserve the natural characteristics (eigenquantities) of the

distributed system during the control action. The effectiveness of natural control has been verified experimentally.<sup>11-12</sup>

The robustness of natural control in the presence of parameter uncertainties is examined in Ref. 13. It is shown there that, with the natural control method, errors in the system parameters cannot destabilize the closed-loop system when modal filters are used in conjunction with an adequate number of sensors. Methods for the extraction of the modal coordinates from the system output also include Luenberger observers<sup>14</sup> and temporal filters.<sup>12</sup> The estimates of the controlled modes obtained by such methods are contaminated by the contributions of the unmodeled modes, a phenomenon known as observation spillover.<sup>6</sup> On the other hand, modal filters are spatial filters used to extract the modal coordinates from the system output without any observation spillover.<sup>8</sup> They make use of the distributed nature of the system and the orthogonality of the eigenfunctions. Robustness with respect to discretization methods is investigated in Ref. 15, where it is concluded that independent controls are robust with respect to discretization by a finite-element-type method. The robustness of other control methods and the sensitivity of distributed systems have been examined in Refs. 16-19.

This paper is concerned with the robustness characteristics of control systems with respect to both parameter and discretization errors and when both observers and modal filters are used. It is shown that if observers are used, regardless of the control method, the separation principle is no longer valid. Therefore, the robustness of such systems cannot be assessed. It is demonstrated here that if natural control is considered in conjunction with modal filters and an adequate number of sensors, a combination of model uncertainties and errors in the eigenvalues and eigenfunctions cannot destabilize the closed-loop system. It is concluded that knowledge of the exact eigenfunctions is not required to control a distributed system; rather a set of admissible functions from a complete set is sufficient. In such cases, however, the modal filters no longer extract the modal coordinates; instead, they extract other coordinates depending on the admissible functions.

One important aspect of robustness is the amount of reduction in the stability margin due to errors and uncertainties. Reference 13 suggests a perturbation analysis to investigate the sensitivity of the closed-loop system. Here, another method, which makes use of Gerschgorin's disks,<sup>20</sup> is proposed to analyze the sensitivity of the open-loop system and the stability margin of the corresponding closed-loop system. A numerical example is presented.

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## II. Equations of Motion

The equations of motion for a self-adjoint distributed-parameter system can be written in the form of the partial differential equation (see Ref. 4, Sec. 5-4)

$$Lu(P,t) + M(P)\partial^2 u(P,t)/\partial t^2 = f(P,t) + w(P,t) \quad (1)$$

which must be satisfied at every point  $P$  of the domain  $D$  and where  $u(P,t)$  is the displacement of point  $P$ ,  $L$  a linear differential self-adjoint operator of order  $2p$  admitting a discrete spectrum,  $M(P)$  the distributed mass,  $f(P,t)$  the distributed controls, and  $w(P,t)$  the actuator noise (generally considered to be a stochastic Gaussian process). The displacement  $u(P,t)$  is subject to the boundary conditions  $B_i u(P,t) = 0$  ( $i = 1, 2, \dots, p$ ), where  $B_i$  are linear differential operators. The solution of the associated eigenvalue problem consists of a denumerably infinite set of eigenvalues  $\lambda_r$  and associated eigenfunctions  $\phi_r$  ( $r = 1, 2, \dots$ ). The eigenvalues are related to the natural frequencies  $\omega_r$  by  $\lambda_r = -\omega_r^2$  ( $r = 1, 2, \dots$ ). The zero frequencies correspond to the translational and rotational rigid-body modes. Because  $L$  is self-adjoint, the natural modes are mutually orthogonal and can be normalized so as to satisfy  $\int_D M \phi_r \phi_s dD = \delta_{rs}$  and  $\int_D \phi_s L \phi_r dD = \lambda_r \delta_{rs}$  ( $r, s = 1, 2, \dots$ ) where  $\delta_{rs}$  is the Kronecker delta.

Using the expansion theorem<sup>4</sup>

$$u(P,t) = \sum_{r=1}^{\infty} \phi_r(P) u_r(t) \quad (2)$$

where  $u_r$  are modal coordinates, Eq. (1) can be replaced by the infinite set of ordinary differential equations

$$\ddot{u}_r(t) + \omega_r^2 u_r(t) = f_r(t) + w_r(t), \quad r = 1, 2, \dots \quad (3)$$

known as modal equations, in which

$$f_r(t) = \int_D \phi_r(P) f(P,t) dD \quad (4a)$$

$$r = 1, 2, \dots$$

$$w_r(t) = \int_D \phi_r(P) w(P,t) dD \quad (4b)$$

are modal control forces and modal excitations due to actuator noise, respectively. The modal control forces are generally determined by a feedback control method, expressed functionally as

$$f_r(t) = f_r[u_1(t), \dot{u}_1(t), u_2(t), \dot{u}_2(t), \dots], \quad r = 1, 2, \dots, n \quad (5)$$

When designing a control system, one assumes that the parameters contained in the equations of motion, as well as the eigenvalues and eigenfunctions associated with the controlled modes, are known accurately. More often than not, these assumptions are not realistic. First, one generally does not have error-free information regarding the parameters contained in the equations of motion. Even if the parameters are known accurately, a closed-form eigensolution is generally not available because of the complexity of the distributed system. In general, a discretization method (such as the Rayleigh-Ritz or finite element) is used to obtain approximations for the eigenvalues and mode shapes.

Let us assume that we have estimates of the mass and stiffness operators and a set of admissible functions that are  $p$  times differentiable and that satisfy the geometric boundary conditions and are orthogonal with respect to the estimate of the mass and stiffness distributions. We will attempt to control the distributed system using this postulated information that we think is the description of the actual system. To this end, we express the motion of the actual system in terms of the generalized coordinates  $v_1(t), v_2(t), \dots$ , of the postulated system of admissible functions  $\psi_r(P)$ ,  $r = 1, 2, \dots$ . Using the

expansion theorem<sup>4</sup>

$$u(P,t) = \sum_{r=1}^{\infty} \psi_r(P) v_r(t) \quad (6)$$

substituting Eq. (6) into Eq. (1), multiplying by  $\psi_s(P)$ , and integrating over the spatial domain, we obtain the equations of motion of the distributed system in terms of  $v_1(t), v_2(t), \dots$ , as

$$M\ddot{v}(t) + K v(t) = \hat{f}(t) + \hat{w}(t) \quad (7)$$

where  $M$  and  $K$  are symmetric matrices of infinite order with entries

$$M_{rs} = \int_D M(P) \psi_r(P) \psi_s(P) dD$$

$$K_{rs} = \int_D \psi_r(P) L \psi_s(P) dD \quad r, s = 1, 2, \dots \quad (8)$$

and

$$v(t) = [v_1(t) \ v_2(t) \ \dots]^T$$

$$\hat{f}(t) = [\hat{f}_1(t) \ \hat{f}_2(t) \ \dots]^T \quad (9)$$

$$\hat{w}(t) = [\hat{w}_1(t) \ \hat{w}_2(t) \ \dots]^T$$

where

$$\hat{f}_r(t) = \int_D \psi_r(P) f(P,t) dD \quad (10a)$$

$$\hat{w}_r(t) = \int_D \psi_r(P) w(P,t) dD \quad (10b)$$

Note that Eqs. (3) and (7) are representations of the same distributed system. However, in the presence of parameter and discretization errors, the modal control forces are chosen as a function of the generalized coordinates such that

$$\hat{f}_r(t) = \hat{f}_r[v_1(t), \dot{v}_1(t), v_2(t), \dot{v}_2(t), \dots], \quad r = 1, 2, \dots, n \quad (11)$$

Next, we wish to examine the cases for which one controls a distributed-parameter system using a set of admissible functions instead of the actual eigenfunctions. The first case is one in which no knowledge is available about the distributed system except for the geometric boundary conditions (i.e., displacement and slope) and the order of the stiffness operator. One can still generate a set of admissible functions.

The second case is one in which the mass and stiffness operators are known, but the natural boundary conditions (shear and moment) are not known and a closed-form solution of the postulated system is available. This solution represents a set of admissible functions for the actual system because the natural boundary conditions are not satisfied. A third case is analyzed in Ref. 13 in which the mass and stiffness distributions are not known, while the nature of the stiffness operator and all of the boundary conditions are known. Here, the closed-form solution, obtained by solving the eigenvalue problem associated with the postulated system, yields a set of comparison functions that constitute a subset of admissible functions. Clearly, the second and third cases do not have widespread applications and are of academic interest only.

The most common use of admissible functions involves the case when a closed-form solution of the differential eigenvalue problem is not available and an approximate eigensolution is obtained by using a finite-element or Galerkin-type discretization method. In addition, an accurate mathematical model may not be available.

In a Galerkin-type method, the motion of the distributed system is expressed in the form<sup>20</sup>

$$u^k(P,t) = \sum_{i=1}^k a_i \beta_i(P) \quad (12)$$

where  $u^k(P, t)$  is the  $k$ th order approximation of  $u(P, t)$ ,  $\beta_i(P)$ , are a set of admissible functions and  $a_i$  undetermined coefficients. If  $\beta_i(P)$  ( $i=1, 2, \dots, k$ ) are defined over the entire domain of the distributed system, the discretization becomes a Rayleigh-Ritz type method. On the other hand, if the admissible functions are local, that is, defined over a portion of the domain, then the discretization method becomes a finite element one. By seeking stationary values of the Rayleigh's quotient, the differential eigenvalue problem is converted into an algebraic one of the form<sup>20</sup>

$$\bar{K}a = \lambda \bar{M}a \quad (13)$$

where  $\bar{M}$  and  $\bar{K}$  are symmetric matrices of order  $k \times k$ , such that

$$\begin{aligned} \bar{K}_{ij} &= \int_D \beta_i(P) \bar{L} \beta_j(P) dD \\ \bar{M}_{ij} &= \int_D \bar{M}(P) \beta_i(P) \beta_j(P) dD \\ a &= [a_1 a_2 \dots a_k]^T \end{aligned} \quad i, j = 1, 2, \dots, k \quad (14)$$

and  $\bar{L}$  and  $\bar{M}(P)$  are estimates of the stiffness and mass operators, respectively.

The solution of the algebraic eigenvalue problem yields a set of eigenvalues that are approximations of the actual eigenvalues and corresponding eigenvectors. The eigenvectors represent approximations of the first  $k$  eigenfunctions  $\phi_r(P)$  ( $r=1, 2, \dots, k$ ) in the form

$$\phi_r^k(P) = \sum_{i=1}^k a_{ir} \beta_i(P), \quad r=1, 2, \dots, k \quad (15)$$

where  $a_{ir}$  is the  $i$ th component of the  $r$ th eigenvector that, in general, is in normalized form with respect to the mass and stiffness matrices  $\bar{M}$  and  $\bar{K}$ . It follows that the approximate eigenfunctions  $\phi_r^k(P)$  are orthogonal with respect to the estimates of the mass and stiffness operators. They also are linear combinations of the admissible functions  $\beta_i(P)$  ( $i=1, 2, \dots, k$ ) so that they are admissible functions themselves.

### III. Control System Design

We consider discrete (in space) actuators to implement the control action. To this end, we can treat the discrete actuator forces  $F_j(t)$  ( $j=1, 2, \dots, m$ ), where  $m$  is the number of actuators, as distributed by writing

$$f(P, t) = \sum_{j=1}^m F_j(t) \delta(P - P_j) \quad (16)$$

Introduction of Eq. (16) into Eqs. (4) and (10a) yields<sup>8</sup>

$$f_r(t) = \sum_{j=1}^m \phi_r(P_j) F_j(t) \quad (17a)$$

$$\hat{f}_r(t) = \sum_{j=1}^m \psi_r(P_j) F_j(t) \quad (17b)$$

or

$$f(t) = BF(t) \quad (18a)$$

$$\hat{f}(t) = B'F(t) \quad (18b)$$

where the notation is obvious.

In modal feedback control laws, either  $f(t)$  or  $F(t)$  is chosen first by the analyst. For linear feedback, if the system parameters are known, the control law is selected, either in the

form

$$f_c(t) = G\hat{u}_c(t) + H\hat{u}_c(t) \quad (19)$$

or

$$F(t) = G'\hat{u}_c(t) + H'\hat{u}_c(t) \quad (20)$$

in which  $BG = G'$  and  $BH = H'$ , where

$$f_c(t) = [f_1(t) f_2(t) \dots f_n(t)]^T \quad (21)$$

is the modal force vector associated with the controlled modes. Depending on the control method, either  $G$  and  $H$  or  $G'$  and  $H'$  are first chosen. The estimate of the modal coordinates, extracted from the system output using Luenberger observers, modal filters, or other methods, can be expressed as

$$\hat{u}_c(t) = [\hat{u}_1(t) \hat{u}_2(t) \dots \hat{u}_n(t)]^T \quad (22)$$

where  $\hat{u}_r(t)$  is the estimate of  $u_r(t)$  ( $r=1, 2, \dots, n$ )

Because of the presence of parameter uncertainties and discretization errors, only  $\hat{v}_r(t)$  ( $r=1, 2, \dots, n$ ) is known, where  $\hat{v}_r(t)$  is the estimate of  $v_r(t)$ . It follows that one resorts to choosing the control gains using

$$\hat{f}_c(t) = G\hat{v}_c(t) + H\hat{v}_c(t) \quad (23)$$

or

$$F(t) = G''\hat{v}_c(t) + H''\hat{v}_c(t) \quad (24)$$

where

$$\begin{aligned} \hat{f}_c(t) &= [\hat{f}_1(t) \hat{f}_2(t) \dots \hat{f}_n(t)]^T \\ \hat{v}_c(t) &= [\hat{v}_1(t) \hat{v}_2(t) \dots \hat{v}_n(t)]^T \end{aligned} \quad (25)$$

and

$$G = B'G'', \quad H = B'H'' \quad (26)$$

It is well known that the residual modes are excited when discrete actuators are used. From Eqs. (4a), (10a), and (17), we obtain

$$f(t) = \begin{bmatrix} G & 0 \\ \hline B_R G' & 0 \end{bmatrix} \hat{u}(t) + \begin{bmatrix} H & 0 \\ \hline B_R H' & 0 \end{bmatrix} \hat{u}(t) \quad (27)$$

where

$$B_R = [B_R]_{ij} = \phi_i(P_j), \quad j=1, 2, \dots, m; \quad i=n+1, n+2, \dots \quad (28)$$

When the generalized coordinates  $v_r(t)$  are used, the modal control vector appears in the form

$$\hat{f}(t) = \begin{bmatrix} G & 0 \\ \hline B_R' G'' & 0 \end{bmatrix} \hat{v}(t) + \begin{bmatrix} H & 0 \\ \hline B_R' H'' & 0 \end{bmatrix} \hat{v}(t) \quad (29)$$

where  $B_{R'ij} = \psi_i(P_j)$  with  $i=n+1, n+2, \dots$  and  $j=1, 2, \dots, m$ . It is clear that the terms  $B_R$  and  $B_R'$  cause control spillover.

### IV. Robustness of Control Systems

We wish to analyze the robustness of a control system whose closed-loop equations can be written by combining Eqs. (7) and (29) in the form

$$M\ddot{v}(t) + K\dot{v}(t) = G^*\hat{v}(t) + H^*\hat{v}(t) + \hat{w}(t) \quad (30)$$

where  $G^*$  and  $H^*$  are obtained from Eq. (29). To this end, we need to analyze methods of extracting the generalized coordinates from the system output. Let us first consider the Luenberger observers. Because a Luenberger observer is designed in the state space, we introduce the state vector  $s(t) = [\dot{v}^T(t) \ v^T(t)]^T$ . Note that we can also express the observer design in terms of the vectors  $u(t)$  and  $\dot{u}(t)$ . Equation (30) can be expressed as

$$\dot{s}(t) = As(t) + \bar{G}\hat{s}(t) + w'(t) \quad (31)$$

where

$$A = \begin{bmatrix} 0 & -M^{-1}K \\ I & 0 \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} H^* & G^* \\ 0 & 0 \end{bmatrix} \quad (32)$$

and where  $w'(t) = [\dot{w}^T(t) \ 0^T]^T$  denotes the contribution of the actuator noise and  $\hat{s}(t)$  the estimate of  $s(t)$ . Note that, in order to obtain Eq. (32) and the further results associated with robustness, we need to consider finite-dimensional models where, in reality, the matrices  $M$ ,  $K$ ,  $G^*$ , and  $H^*$  are infinite-dimensional. However, the results in this paper are valid regardless of the order of the system we choose. Thus, the results and proofs of stability presented in this paper must be regarded as heuristic.

The Luenberger observer has the form

$$\dot{\hat{s}}(t) = (A_1 + \bar{G})\hat{s}(t) + K'[y(t) - \hat{y}(t)] \quad (33)$$

where

$$A_1 = \begin{bmatrix} 0 & -\Lambda \\ I & 0 \end{bmatrix} \quad (34)$$

is the state matrix we think we have and  $\Lambda$  a diagonal matrix containing the eigenvalues  $\bar{\lambda}_j$  obtained from the relationship  $\bar{\lambda}_j = \int_D \psi_j(P) \bar{L} \psi_j(P) dD$ , ( $j=1,2,\dots$ ) and where  $K'$  is the observer gain matrix chosen by the analyst and  $y(t)$  the observation vector, such that

$$y(t) = Ds(t), \quad \hat{y}(t) = D\hat{s}(t) \quad (35)$$

where the entries of  $D$  depend on the sensor locations. Note that, if there are no parameter or discretization errors,  $A = A_1$ . Introducing the error vector  $e(t) = s(t) - \hat{s}(t)$ , we can combine Eqs. (31) and (33) into

$$\begin{bmatrix} \dot{s}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A + \bar{G} & -\bar{G} \\ A - A_1 & A_1 - K'D \end{bmatrix} \begin{bmatrix} s(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} w'(t) \\ w'(t) \end{bmatrix} \quad (36)$$

It is clear that because of parameter and discretization errors, which give rise to the term  $(A - A_1)$  in Eq. (36), the separation principle is no longer valid and the closed-loop poles of both the observer and control system cannot be assessed qualitatively or obtained independently of each other.

Next, we consider modal filters, which are spatial filters and take advantage of the orthogonality of the modes.<sup>8,21</sup> Use of modal filters is based on the second part of the expansion theorem,<sup>4</sup> which states

$$\begin{aligned} u_r(t) &= \int_D M(P) \phi_r(P) u(P, t) dD \\ \dot{u}_r(t) &= \int_D M(P) \phi_r(P) \dot{u}(P, t) dD \end{aligned} \quad r=1,2,\dots \quad (37)$$

If measurements of the system displacement (and velocity) are available along every point on the distributed system, one can use Eqs. (37) to determine  $u_r(t)$  and  $\dot{u}_r(t)$  exactly. The problem is to generate  $u_r(t)$  and  $\dot{u}_r(t)$  when the measurements of only a few points are available. The implementation of modal

filters with discrete sensors requires interpolation of the discrete measurements to yield the continuous displacement and velocity profiles. The choice of the interpolation functions is extremely important in determining the accuracy and computational effort required in implementing modal filters. Note that Eq. (37) can be written in terms of the generalized coordinates  $v_r(t)$  as

$$\begin{aligned} v_r(t) &= \int_D \bar{M}(P) \psi_r(P) u(P, t) dD \\ \dot{v}_r(t) &= \int_D \bar{M}(P) \psi_r(P) \dot{u}(P, t) dD \end{aligned} \quad r=1,2,\dots \quad (38)$$

We propose to choose an interpolation process so that the estimated distributed profile  $\hat{u}(P, t)$  can be described as

$$\hat{u}(P, t) = \sum_{j=1}^k C(P, P_j) [u(P_j, t) + q_j(t)] \quad (39)$$

where  $k$  is the number of sensors,  $C(P, P_j)$  the interpolation functions, and  $q_j(t)$  ( $j=1,2,\dots,k$ ) the measurement noise associated with the  $j$ th sensor. Note that we took an interpolation process where the time and space dependencies of the interpolation are separate. The advantage of this becomes evident when we introduce Eq. (39) into Eq. (38), which yields

$$\hat{v}_r(t) = \sum_{j=1}^k C'_{rj} [u(P_j, t) + q_j(t)] \quad (40)$$

and

$$C'_{rj} = \int_D \bar{M}(P) \psi_r(P) C(P, P_j) dD, \quad r=1,2,\dots; \quad j=1,2,\dots,k \quad (41)$$

These are modal filter gains and can be computed off-line, before the control process begins, so that extraction of the modal coordinates reduces to a matrix multiplication during on-line control. Among the interpolations of Eq. (39) are finite element or Rayleigh-Ritz types of expansions.<sup>21</sup> The accuracy of modal filters improves as the number of sensors is increased.<sup>8,21</sup> The relationship between  $v(t)$  and  $\hat{v}(t)$  can be expressed as<sup>21</sup>

$$\hat{v}(t) = C'D'v(t) + C'q(t) \quad (42)$$

where  $C'$  is defined by Eqs. (41),  $D'_{ij} = \psi_j(P_i)$ , and  $q(t)$  is the sensor noise vector in the form

$$q(t) = [q_1(t) q_2(t) \dots q_k(t)]^T \quad (43)$$

When a sufficient number of sensors are used,  $C'D'$  approaches an identity matrix, at least for the controlled modes, thus eliminating observation spillover. We can partition  $C'D'$  into

$$C'D' = \begin{bmatrix} O_{c1} & O_{c2} \\ O_{R1} & O_{R2} \end{bmatrix} \quad (44)$$

where the partitions separate the controlled and residual modes. When a sufficient number of sensors is available  $O_{c1} = 1$  and  $O_{c2} = 0$ . Note that we need to extract only the controlled modes from the system output during the control process. Methods for determination of the sufficient number of sensors are outlined in Refs. 8 and 21. It is also shown in Ref. 21 that the effects of sensor noise become less critical as the number of sensors is increased.

Next, let us analyze the stability of linear feedback control systems in conjunction with modal filters. Introducing Eq. (42) into Eq. (30), multiplying by  $v^T(t)$ , and integrating over time, we obtain

$$\begin{aligned}
& [\dot{v}^T(t)M\dot{v}(t) + v^T(t)Kv(t)] \Big|_0^{t_f} \\
&= \int_0^{t_f} \dot{v}^T(t)H^*C'D'\dot{v}(t)dt + \int_0^{t_f} v^T(t)G^*C'D'v(t)dt \\
&+ \int_0^{t_f} v^T(t)[G^*C'q(t) + H^*C'\dot{q}(t) + \hat{w}(t)]dt \quad (45)
\end{aligned}$$

where  $t_f$  is the final time. Note that in obtaining Eq. (45) we made use of the fact that  $M$  and  $K$  are symmetric matrices. The left side of Eq. (45) denotes the difference in energy between the final and initial states, or  $E(t_f) - E(0)$ . The last term on the right side of Eq. (45) is a random variable because it is a linear combination of  $q(t)$  and  $\hat{w}(t)$ , which are themselves random variables. Considering Eqs. (29), (30), and (44) we obtain

$$G^*C'D' = \begin{bmatrix} GO_{c1} & GO_{c2} \\ B_R'G''O_{c1} & B_R'G''O_{c2} \end{bmatrix} \quad (46)$$

$$H^*C'D' = \begin{bmatrix} HO_{c1} & HO_{c2} \\ B_R'H''O_{c1} & B_R'H''O_{c2} \end{bmatrix}$$

which shows that errors in estimation of the residual modes have no effect on the system energy.

For stability the energy in the control system, i.e., the right side of Eq. (45), must decay asymptotically. The last term on the right side of Eq. (45) is a random variable. Therefore, it has a random effect on the system energy and does not affect the system stability. The decay in the system energy is possible when  $H^*C'D$  and  $G^*C'D$  are symmetric negative-definite matrices. Clearly, when a sufficient number of sensors is not available, it is not possible to have such a situation. However, assuming a sufficient number of sensors (a number determined by one of the guidelines in Refs. 8 and 21), we can write  $O_{c1} = 1$ ,  $O_{c2} = 0$ . Note that by comparing  $O_{c1}$  and  $O_{c2}$  for different numbers of sensors, one can determine the sufficient number of sensors for a specific control problem.<sup>21</sup> Under such circumstances, the sign definiteness of the first two integrals on the right side of Eq. (45) is determined by  $G$  and  $H$ . Therefore, we conclude that, if modal filters are used in conjunction with a sufficient number of sensors and if the control gain matrices  $G$  and  $H$  are chosen such that they are symmetric and negative definite, the control system is always stable in the presence of parameter and discretization errors.

Next, let us investigate existing control methods where the control gain matrices are known to be symmetric and negative semidefinite. One such method is colocated control,<sup>19</sup> where the actuators are colocated with the sensors. The stability characteristics of colocated control are well known. Indeed, it is easy to show that use of colocated control leads to a negative definite  $H$  matrix, which insures stability. An advantage of colocated control over other methods is that it does not require the use of an observer or modal filter. Another control method where the gain matrices are sign definite is natural control, also known as independent modal-space control (IMSC). In natural control, the modal control forces are selected first and the modal control force associated with each mode depends only on that mode, so that we can write

$$f_r(t) = f_r[u_r(t), \dot{u}_r(t)], \quad \hat{f}_r(t) = \hat{f}_r[v_r(t), \dot{v}_r(t)] \quad r = 1, 2, \dots, n \quad (47)$$

In the case of linear proportional control and when parameter and discretization errors are present, the control law becomes

$$\hat{f}_r(t) = g_r \hat{v}_r(t) + h_r \hat{v}_r(t), \quad r = 1, 2, \dots, n \quad (48)$$

where  $g_r$  and  $h_r$  are control gain parameters. It is clear that the control gain matrices  $G$  and  $H$  are diagonal. Obviously,  $H$  is designed to be negative definite and  $G$  can be chosen as negative definite. For optimal control,  $G$  is negative definite.<sup>13</sup>

Synthesis of the actual control forces is possible by inverting Eq. (18b), which requires  $B'$  to be square and nonsingular. This is possible when the number of actuators is equal to the number of controlled modes. Another method of obtaining the actuator forces is given in Ref. 10. For the hypothetical case of distributed controls, we can make use of the second part of the expansion theorem and Eqs. (4a) and (10a) to write

$$\begin{aligned}
f(P, t) &= M(P) \sum_{r=1}^n \phi_r(P) f_r(t) \\
f(P, t) &= \bar{M}(P) \sum_{r=1}^n \psi_r(P) \hat{f}_r(t) \quad (49)
\end{aligned}$$

which can conveniently be expressed in terms of the gain operators  $G$  and  $H$ , such that

$$f(P, t) = Gu(P, t) + H\dot{u}(P, t) \quad (50)$$

It can easily be shown that when the system eigenfunctions are known<sup>10</sup>

$$\int_D \phi_r(P) G \phi_s(P) dD = g_r \delta_{rs}, \quad \int_D \phi_r(P) H \phi_s(P) dD = h_r \delta_{rs} \quad (51)$$

and when the system eigenfunctions  $\phi_r(P)$  are not known and the admissible functions  $\psi_r(P)$  are used in the control design,

$$\int_D \psi_r(P) G \psi_s(P) dD = g_r \delta_{rs}, \quad \int_D \psi_r(P) H \psi_s(P) dD = h_r \delta_{rs} \quad (52)$$

Various features unique to natural controls are discussed in Ref. 3. It is further concluded here that if *natural control is used in conjunction with modal filters and an adequate number of sensors, the control system is guaranteed to be stable in the presence of parameter uncertainties and errors due to spatial discretization*. With this in mind, we confine further discussions in this paper to natural control in conjunction with modal filters and negligible observation spillover.

## V. The Degree of Robustness

In the preceding section, it was shown that one can control a distributed system using natural control and a set of admissible functions. As mentioned in Sec. II, one can generate a set of admissible functions by knowing only the dimensions and geometric boundary conditions of the distributed system. However, even though the closed-loop system is guaranteed to be stable, the actual closed-loop poles may deviate from the desired ones substantially, if the admissible functions do not resemble the eigenfunctions, which results in degradation of the control system performance. It is of interest to examine the extent of deviation from the desired closed-loop system and to investigate the factors affecting the stability margin.

First, let us analyze the effect of structural damping on the stability margin. In the presence of damping, the equation (1) of motion looks like

$$Lu(P, t) + C\dot{u}(P, t) + M(P)\ddot{u}(P, t) = f(P, t) + w(P, t) \quad (53)$$

where  $D$  is the damping operator. Introducing Eqs. (6) into Eq. (53) and repeating the same procedure to derive Eq. (45), we obtain in the case of a sufficient number of sensors

$$\begin{aligned}
& [\dot{v}^T(t)M\dot{v}(t) + v^T(t)Kv(t)] \Big|_0^{t_f} \\
&= \int_0^{t_f} \dot{v}^T(t)H^*C'D'\dot{v}(t)dt + \int_0^{t_f} v^T(t)G^*C'D'v(t)dt \\
&- \int_0^{t_f} \dot{v}^T(t)(D\dot{v}(t))dt + R(t_f) \quad (54)
\end{aligned}$$

The last term on the right side of Eq. (3) denotes the effects of actuator and sensor noise. The third term on the right of Eq. (54) represents the effect of damping on the change of energy in the system. The matrix  $D$  has entries

$$D_{ij} = \int_D \psi_i(P) C \psi_j(P) dD, \quad i, j = 1, 2, \dots \quad (55)$$

Consider the nature of the damping operator. In general,  $C$  is assumed to be linear, self-adjoint, and positive semidefinite. The matrix  $D$  then becomes symmetric and positive semidefinite, which makes the last term on the right side of Eq. (54) dissipative. Therefore, the structural damping still adds to the stability margin when the control forces are designed based on a set of admissible functions, instead of the actual eigenfunctions.

Next, we wish to analyze those stability bounds of the closed-loop poles in the actual systems that directly affect the control system performance. In the case of linear proportional control, the closed-loop equations are given by Eq. (30). One can then select various sets of admissible functions, solve the eigenvalue problem associated with Eq. (30), and compare the closed-loop poles. Reference 13 proposes a matrix perturbation analysis to examine the effects of having an incorrect model when the errors in the system parameters are not very large. Here we propose a qualitative and more general analysis that is not restricted to small errors in the system parameters. To this end, we wish to express the system equations in the state space in the form

$$\begin{bmatrix} \ddot{u}(P, t) \\ \dot{u}(P, t) \end{bmatrix} = \begin{bmatrix} 0 & -M^{-1}L \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u(P, t) \\ \dot{u}(P, t) \end{bmatrix} + \begin{bmatrix} f(P, t) \\ 0 \end{bmatrix} \quad (56)$$

Given the postulated (estimated) mass distribution  $\bar{M}(P)$  and stiffness operator  $\bar{L}(P)$ , the associated eigenvalue problem (in the state space) yields

$$\begin{bmatrix} \dot{u}(P, t) \\ u(P, t) \end{bmatrix} = \sum_{r=1}^{\infty} \begin{bmatrix} \psi_{1r}(P) \\ \psi_{2r}(P) \end{bmatrix} v_r(t) \quad (57)$$

$$\begin{bmatrix} \psi_{1r}(P) \\ \psi_{2r}(P) \end{bmatrix} = \begin{bmatrix} i\bar{\omega}_r \psi_r(P) \\ \psi_r(P) \end{bmatrix}$$

where  $\bar{\omega}_r$  are the natural frequencies of the postulated system and  $\bar{\omega}_r^2 = -1$ . Substitution of Eqs. (57) into Eq. (56) yields

$$\dot{v}(t) = \bar{L}v(t) + \bar{f}(t) \quad (58)$$

where  $v(t)$  is the state vector (different from its earlier definition) and

$$\bar{L}_{rs} = \int_D \left[ \frac{1}{2i\bar{\omega}_r} \psi_r(P) - \frac{1}{2} \psi_r(P) \right] \begin{bmatrix} 0 & M^{-1}L \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_{1s}(P) \\ \psi_{2s}(P) \end{bmatrix} dD \quad (59)$$

When natural control is used, the control vector  $\bar{f}(t)$  has the form

$$\bar{f}(t) = G'v(t) \quad (60)$$

where

$$G'_{rs} =$$

$$\int_D \left[ \frac{1}{2i\bar{\omega}_r} \psi_r(P) - \frac{1}{2} \psi_r(P) \right] \begin{bmatrix} \bar{M}^{-1}H & \bar{M}^{-1}G \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_{1s}(P) \\ \psi_{2s}(P) \end{bmatrix} dD \quad (61)$$

where  $G$  and  $H$  are gain operators. Note that  $G'_{rs}$  is diagonalizable only in the case of natural control and if the mass operator is known. When  $G$  and  $H$  are chosen according to Eqs. (50-52) and  $\bar{M} = M$ , we obtain

$$G'_{rs} = g_r \delta_{rs}, \quad r, s = 1, 2, \dots \quad (62)$$

The eigenvalue problem for the closed-loop system can be obtained by introducing Eqs. (60) into Eq. (57), which yields

$$Av = \lambda v \quad (63)$$

where  $A = \bar{L} + G'$ . According to the first theorem of Gerschgorin,<sup>20</sup> the eigenvalues of  $A$  lie in the circles with centers  $c_r = A_{rr}$  and associated radii  $R_r$ , where

$$R_r = \sum_{\substack{s=1 \\ r \neq s}}^{\infty} |A_{rs}|, \quad r, s = 1, 2, \dots \quad (64)$$

Note that if the system parameters are known, in the uncontrolled case, Gerschgorin's disks represent points. Considering the definition of  $c_r$ ,  $R_r$ , and  $A$ , we obtain for natural control

$$c_r = A_{rr} = \bar{L}_{rr} + G'_{rr} = \bar{L}_{rr} + g_r$$

$$R_r = \sum_{\substack{s=1 \\ r \neq s}}^{\infty} |A_{rs}| = \sum_{\substack{s=1 \\ r \neq s}}^{\infty} |\bar{L}_{rs} + G'_{rs}| = \sum_{\substack{s=1 \\ r \neq s}}^{\infty} |\bar{L}_{rs}| \quad (65)$$

It is clear from Eqs. (65) that the addition of controls only shifts the center of Gerschgorin's disks and does not alter the radii of the open-loop system when natural control is used. We can then conclude that, *for natural control with a known mass distribution, the robustness of the uncontrolled system is identical to the robustness of the closed-loop system*. Note that for other methods,  $G'$  is not diagonal, so that the radii of Gerschgorin's disks are altered with the addition of controls, which is an indication that the robustness of the control system deteriorates for methods other than natural control. It can easily be shown that in the case of finite-dimensional control, the above results are valid for the robustness of the closed-loop system for the controlled modes. Intuitively, one expects the stability margin to improve as the admissible functions come to resemble the actual eigenfunctions, because the nature of the admissible functions directly affects the off-diagonal terms of matrix  $A$ . It should also be noted, that, while the stability is guaranteed when natural control is used, the performance of the control system deteriorates as the admissible functions look less like the eigenfunctions. Therefore, in order to determine the level of error that can be tolerated, a performance sensitivity analysis should be carried out as well.

## VI. Illustrative Example

Let us consider the problem of controlling the axial vibration of a tapered bar of length  $\ell = 10$  and fixed at one end. The mass and stiffness distributions have the form

$$M(x) = 2(1 - x/\ell), \quad EA(x) = 2(1 - x/\ell) \quad (66)$$

The stiffness operator is

$$L = -\frac{d}{dx} \left[ EA(x) \frac{d}{dx} \right] \quad (67)$$

and the boundary conditions are  $B_1(0) = 1$ ,  $B_1(\ell) = EA(\ell)(d/dx)$ . This model has been chosen because the associated eigenvalue problem lends itself to a closed-form solution, such that the performance of the control system can be compared for different sets of admissible functions. The transcendental equation associated with the eigenvalue pro-

blem is given by<sup>3</sup>

$$J_0(\ell\omega) = 0 \quad (68)$$

where  $J_0$  is the Bessel function of order zero and of the first kind and  $\omega$  the natural frequency. The solution of Eq. (68) can readily be found numerically. The first six natural frequencies of the bar are 0.24048, 0.55201, 0.86537, 1.1792, 1.4931, and 1.8071 rad/s.

The normalized eigenfunctions have the form<sup>3</sup>

$$\phi_r(x) = \frac{J_0[\omega_r(\ell-x)]}{\sqrt{\ell} J_1(\ell\omega_r)}, \quad r=1,2,\dots \quad (69)$$

where  $J_1$  is the Bessel function of the first order and of the first kind.

Let us consider linear proportional control and select the modal control gains as

$$h_r = -0.3, \quad g_r = 0, \quad r=1,2,\dots,n \quad (70)$$

and  $n=6$  denotes the number of controlled modes. It is easy to show that, for natural control, the closed-loop poles of each mode are given by

$$\lambda_r = h_r/2 \pm \sqrt{(h_r/2)^2 - \omega_r^2 + g_r}, \quad r=1,2,\dots,6 \quad (71)$$

Next, let us implement natural control, assuming that our postulated system has as eigenfunctions

$$\psi_r(x) = \frac{2}{\sqrt{\ell}} \sin \left[ (2r-1) \frac{\pi x}{2\ell} \right], \quad r=1,2,\dots,6 \quad (72)$$

We note that this series of admissible functions are the eigenfunctions of a uniform bar<sup>4</sup> with a mass distribution of  $M(x) = 1/2$ . Indeed, using Eqs. (72) as admissible functions is the equivalent to having errors in the mass and stiffness distributions. As a second set of admissible functions, let us select simple polynomials in the form

$$\psi_r(x) = (x/\ell)^r, \quad r=1,2,\dots,6 \quad (73)$$

Table 1 compares the closed-loop poles of the actual system when the admissible functions are chosen according to Eqs. (66) and (67). It is clear that the closed-loop poles deviate from their desired values when the admissible functions are different from the actual eigenfunctions. Taking the admissible functions as simple polynomials is a very crude approximation, so that the closed-loop poles in this case are not close to the desired ones at all, which implies a very large degradation in the system performance. On the other hand, the eigenfunctions of the uniform bar resemble the eigenfunctions of the tapered bar to be controlled, so that the closed-loop poles look more like the actual ones. In both cases, however, the imaginary parts of the closed-loop poles are quite different from the desired imaginary parts. This is an indication that part of the control energy is used to change the natural characteristics of the control system and to recouple the closed-loop poles. Changing these natural characteristics serves no useful pur-

pose. Also, such a change uneasily expends some of the control energy. Obviously, this is the consequence of having a control system based on an erroneous model.

The results described in this paper are in agreement with the results in Ref. 13. There, the closed-loop behavior is investigated for a system whose estimated mass and stiffness operators differ very little from their actual values (Ref. 13, Table 1). The actual closed-loop poles there are very close to the desired ones and corroborate the results in this paper that, as the estimated mass and stiffness properties and hence the associated admissible functions come to resemble the actual model parameters and eigensolution, the closed-loop system approaches the desired one.

It should be noted that the eigenvalue analysis performed above is based on some sort of truncation. Because the actual system is infinite-dimensional, an exact eigenvalue analysis is not possible. However, a good approximation can be made by taking a sufficiently large-order model, so that the truncated modes will have insignificant contributions to the system response. In this case, we chose to truncate the residual modes, knowing that they are guaranteed to be stable because natural control is used in conjunction with the modal filters. Finally, it should be emphasized that a stability analysis alone is not sufficient to assess the behavior of a control system. A performance sensitivity analysis should also be conducted. However, this performance sensitivity is problem dependent.

## VII. Conclusions

The effects of discretization errors and parameter uncertainties on the closed-loop behavior of distributed systems are analyzed. It is shown that having discretization errors, together with errors in the parameters that enter into the equations of motion, is the equivalent to controlling the distributed system by a set of admissible functions instead of the actual eigenfunctions. Such an approach will always result in a stable closed-loop system when natural control is used in conjunction with modal filters and an adequate number of sensors. The stability margin becomes smaller as the admissible functions tend not to resemble the actual eigenfunctions. A robustness analysis using Gerschgorin's disks indicates that the deviation of the eigenvalues is identical for both the open- and closed-loop systems, provided natural control is used.

## References

- Simon, J.D. and Mitter, K., "A Theory of Modal Control," *Information and Control*, Vol. 13, 1968, pp. 316-353.
- Porter, B. and Crossley, T.R., *Modal Control—Theory and Applications*, Taylor and Francis Ltd., London, 1972.
- Meirovitch, L., Baruh, H., and Oz, H., "A Comparison of Control Techniques for Large Flexible Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 6, No. 4, 1983, pp. 302-310.
- Meirovitch, L., *Analytical Methods in Vibrations*, MacMillan Co., New York, 1967.
- Strang, G. and Fix, G.J., *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, N.J., 1973.
- Balas, M.J., "Active Control of Flexible Systems," *Journal of Optimization Theory and Applications*, Vol. 25, No. 3, 1978, pp. 415-436.
- Skelton, R.E. and Gregory, C.A., "Measurement Feedback and Model Reduction by Modal Cost Analysis," *Proceedings of the JACC*, 1979, pp. 211-219.
- Meirovitch, L. and Baruh, H., "Control of Self-Adjoint Distributed-Parameter Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 5, Jan.-Feb. 1982, pp. 60-66.
- Meirovitch, L. and Oz, H., "Modal-Space Control of Distributed Gyroscopic Systems," *Journal of Guidance and Control*, Vol. 3, March-April 1980, pp. 140-150.
- Meirovitch, L. and Silverberg, L.M., "Globally Optimal Control of Self-Adjoint Distributed Systems," *Optimal Control Applications and Methods*, Vol. 4, 1983, pp. 365-386.
- Meirovitch, L., Baruh, H., Montgomery, R.C., and Williams, J.P., "Nonlinear Natural Control of an Experimental Beam," *Journal of Guidance, Control, and Dynamics*, Vol. 7, No. 4, 1984, pp. 437-442.

Table 1 Closed-loop poles for different admissible functions

Polynomials	Functions used	
	Uniform bar solution	Eigenfunctions
-0.0187 ± 0.2421	-0.0681 ± 0.2358	-0.1500 ± 0.1880
-0.1520 ± 0.7117	-0.0839 ± 0.5484	-0.1500 ± 0.5312
-0.0002, -0.0004	-0.0929 ± 0.8605	-0.1500 ± 0.8523
-11806.5, -90876	-0.0982 ± 1.1710	-0.1500 ± 1.1696
-183.21, -5.0296	-0.1021 ± 1.4773	-0.1500 ± 1.4956
-0.0187, -0.0081	-0.0883 ± 1.7817	-0.1500 ± 1.8009

<sup>12</sup>Hallauer, W.L., Skidmore, G.R., and Mesquita, L.C., "Experimental-Theoretical Study of Active Vibration Control," *Proceedings of the International Modal Analysis Conference*, Nov. 1982, pp. 39-45.

<sup>13</sup>Meirovitch, L. and Baruh, H., "Robustness of the IMSC Method," *Journal of Guidance, Control, and Dynamics*, Vol. 6, Jan.-Feb. 1983, pp. 20-25.

<sup>14</sup>Brogan, W.L., *Modern Control Theory*, QPI Publishers, New York, 1974.

<sup>15</sup>Hale, A.L. and Rahn, G.A., "Robust Control of Self-Adjoint Distributed-Parameters Structures," *Journal of Guidance, Control, and Dynamics*, Vol. 7, May-June 1984, pp. 265-273.

<sup>16</sup>Chen, M.J. and Desoer, C.A., "Necessary and Sufficient Condition for Robust Stability of Linear Distributed Feedback Systems," *International Journal of Control*, Vol. 35, No. 2, 1982, pp. 255-267.

<sup>17</sup>Sezer, M.E. and Siljak, D.D., "Robustness of Suboptimal Control: Gain and Phase Margin," *IEEE Transactions on Automatic Control*, Vol. AC-26, No. 4, 1981, pp. 907-912.

<sup>18</sup>Schaechter, D.B., "Closed-Loop Control Performance Sensitivity to Parameter Variations," *Journal of Guidance, Control, and Dynamics*, Vol. 6, Sept.-Oct. 1983, pp. 399-402.

<sup>19</sup>Aubrun, J.N., "Theory of the Control of Structures by Low-Authority Controllers," *Journal of Guidance and Control*, Vol. 3, Sept.-Oct. 1980, pp. 444-451.

<sup>20</sup>Meirovitch, L., *Computational Methods in Structural Dynamics*, Sijthoff-Noordhoff Co., the Netherlands, 1980.

<sup>21</sup>Meirovitch, L. and Baruh, H., "On the Implementation of Modal Filters for Control of Structures," *Proceedings of the AIAA 1984 Guidance and Control Conference*, AIAA, New York, Aug. 1984, pp. 624-636. To appear in *Journal of Guidance, Control, and Dynamics*.

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